

## AN ESTIMATION OF HYDRODYNAMIC ENTRANCE LENGTH FOR FLOW THROUGH CIRCULAR ANNULUS

Arunabha Chanda\* and Nirmalya Bayal\*\*

\*Department of Mechanical Engineering,  
Jadavpur University, Kolkata-700032, India.

\*\*Department of Mechanical Engineering,  
Asansol Engineering College, Asansol-713304, India.

**Abstract** The flow of fluids through ducts has long fascinated engineers, scientists and mathematicians. There are a plethora of situations where a clear understanding of this category of flow is necessary. This work involves an exhaustive study of the hydrodynamically developing flow through annular ducts. A numerical model is developed with the stream function vorticity formulation of the momentum equation. The advantage of the stream function vorticity form lies in the fact that the pressure term is eliminated from the formulation.

### INTRODUCTION

Flow through the entrance region of pipes of different cross sections has immense industrial significance. A lot of work has been carried out both experimentally as well as theoretically for this category of flow. In general four different methods have been applied to solve the developing flow in the hydrodynamic entrance region. Van Dyke lists these as (i) Numerical finite difference solution of the boundary layer equations (ii) linearization of the nonlinear inertia terms (iii) Integral methods and (iv) series expansion, and comments on the assumptions common to the two methods in so far as accounting for the displacement thickness is concerned. However most of the available literature deals with circular cross sections only. Annuli of circular shapes play a crucial role in understanding flow and heat transfer through circular tube heat exchanger. The hydrodynamic entry length problem was analyzed first by Boussinesq by considering a perturbation about the fully developed Poiseuille profile. His infinite series solution was fairly adequate downstream, but poor near the entrance. Atkinson and Goldstein presented the stream function by a power series to find a solution for axial position close to the inlet for circular ducts. They then joined this solution with the one obtained by Boussinesq's method to obtain a velocity distribution in the entire entrance region. Schmidt and Zeldin [1] used a finite difference scheme for circular pipes and also have presented a useful review on the topic. Brandit and Gillis [2] have carried out only low Reynolds number analysis without accounting for upstream effects. A proper accounting of upstream influences requires that the flow fields upstream and downstream of the duct

inlet be solved simultaneously. This coupled problems has been analyzed in two specific cases. The pressure drop sustained by creeping (i.e. inertia free) flow was examined by Weissberg [3] for the case of a circular tube to which fluid is supplied from an unbounded upstream space. Wang and Longwell [4] and Van Dyke modeled the upstream region of a parallel plate as an extension of the duct itself. Vrentas et al. [5] and Ursin et al. [6] also carried out a similar analysis and replaced the no slip boundary condition with zero shear condition. This so-called stream tube model does not include the changes in flow cross-section and the rapidly converging streamlines which are frequently encountered just upstream of the inlet. A close study of all the above listed works make it clear that in spite of the voluminous literature available on the topic of flow of fluid through ducts, there exists some areas which have either not been explored at all or no clearly mentioned features are presented. The point where the boundary layers coalesce and the point where the hydrodynamic entrance region ends has been obtained by few researchers for circular ducts but this has not been evaluated for concentric annular ducts.

### GOVERNING EQUATIONS

The governing equations for flow through a circular duct have been solved in the vorticity-transport form. The relevant equations are:

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial r} = -\frac{\omega v}{r} + \nu \left[ \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} \right] \quad \dots(1.1)$$

\*E-mail arunabhachanda@yahoo.com

\*E-mail n\_bayal@rediffmail.com

where  $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}$

The stream functions are defined as:

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad \text{and} \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x} \quad \dots(1.2)$$

These equations are non-dimensionalized with the help of average velocity  $u_0$  and the radius of the duct  $R$

$$u' = \frac{u}{u_0}, \quad v' = \frac{v}{u_0}, \quad x' = \frac{x}{R}, \quad \psi' = \frac{\psi}{u_0 R^2} \quad \text{and} \quad \omega' = \frac{\omega R}{u_0}$$

The non-dimensional parameter Reynolds number is defined as

$$Re = \frac{2u_0 R}{\nu}$$

Thus the governing equations for flow through a circular duct turns out to be:

$$r' \frac{\partial^2 \psi'}{\partial r'^2} - \frac{\partial \psi'}{\partial r'} + r' \frac{\partial^2 \psi'}{\partial x'^2} = \omega' r'^2 \quad \dots(2.1)$$

and 
$$u' \frac{\partial \omega'}{\partial r'} + v' \frac{\partial \omega'}{\partial x'} = -\frac{u_0^2}{R^2} \frac{\omega' v'}{r'} + \frac{1}{Re} \left[ \frac{\partial^2 \omega'}{\partial x'^2} + \frac{\partial^2 \omega'}{\partial r'^2} + \frac{1}{r'} \frac{\partial \omega'}{\partial r'} - \frac{\omega'}{r'^2} \right] \quad \dots(2.2)$$

where  $u' = \frac{1}{r'} \frac{\partial \psi'}{\partial r'}$  and  $v' = -\frac{1}{r'} \frac{\partial \psi'}{\partial x'}$

The flow being axisymmetric, the radial component of velocity on the axis is zero and therefore the axis represents a streamline. Without loss of generality the value of stream function along this streamline has been assigned a value of zero, i.e.  $\psi(0,x)=0$ . The variation of radial velocity along the axis can be alternatively represented by  $\frac{\partial v}{\partial x} = 0$ . The axisymmetric flow also demands that the axial component of velocity  $u$  is maximum on the axis and so  $\frac{\partial v}{\partial r} = 0$ . Using these two expressions, it may be concluded that,  $\omega(0,x)=0$ .

At inlet a uniform velocity profile is applied. On non-dimensionalising, the value of the inlet velocity turns out to be unity, i.e.,

$$u(1,r)=1 \quad \text{and} \quad v(1,r)=0 \quad \dots(3.1)$$

Making use of the above expressions one obtains

$$\psi(1,r) = \int_0^r r dr$$

or, 
$$\psi(1,r) = \frac{1}{2} r^2 \quad \dots(3.2)$$

On account of assumed uniform velocity conditions at the inlet, the vorticity is  $\omega(1,r) = 0$

On the solid wall, fluid velocity is zero on account of no-slip and surface impregnability conditions. So in this case,

$$u(x,1) = v(x,1) = 0 \quad \dots(3.3)$$

Since any solid boundary happens to be a streamline, it is obvious that,  $\psi(x,1) = 0.5$

By virtue of the radial velocity along the wall being zero,

$$\omega(x,1) = -\frac{\partial u}{\partial r} = -\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} = -\frac{\partial^2 \psi}{\partial r^2} \quad \dots(3.4)$$

To formulate the exit boundary conditions, the concept of fully developed flow is exploited. In accordance with it, the stream function and vorticity can be considered to have compared their development beyond the entry length. Therefore at exit of a long pipe of length  $L$ ,

$$\frac{\partial \psi(L,r)}{\partial x} = \frac{\partial \omega(L,r)}{\partial x} = 0 \quad \dots(3.5)$$

Boundary conditions for inner wall is,

$$\omega(x,k) = -\frac{1}{k} \frac{\partial^2 \psi}{\partial r^2} \quad \dots(3.6)$$

### SOLUTION METHODOLOGY

The solution of the above partial differential equations was obtained using a finite difference numerical scheme, which employs a first order up winding technique. The unwinding was done only in the axial direction. This is because the restrictive condition of  $|\Delta x Re| \leq 2$  and  $|\nu \Delta r Re| \leq 2$  are quite insignificant while the radial velocity is considered. Image point technique is used for the wall boundary conditions. The governing equations along with the boundary conditions are solved using a numerical code written in Fortran language to get stream function, vorticity and velocity at every internal grid point. Gauss-Siedel iterative technique is employed for this purpose. The solution procedure starts by applying the initial guesses of stream and vorticity functions and obtain a converged solution by iteration. At first the flow domain is solved for different Reynolds number and the entrance length is obtained. Entrance length is taken to be that length from the entry where the centerline velocity becomes 99% of that in the fully developed zone. To identify the region where the boundary layers coalesce, the edge of the boundary layer is located. Beyond this edge, the shear stress is zero, i.e. the flow is inviscid. For the fully developed profile only on the duct axis, the shear stress is zero and the width of the inviscid zone is actually zero. However in engineering calculations the edge of the boundary layer is determined on the basis of where the velocity reaches 99% of the inviscid flow value. Following this approach, instead of a zero value, the extent of inviscid zone is determined as a non-dimensional radius 0.1 even for the fully developed region. It is therefore, evident that a value of 0.1 for the

radius of inviscid core has been prescribed to find out the location of the boundary layer merger evaluated on the basis of 99% of the axial velocity.

## RESULTS

The numerical code which has been developed using the finite-difference technique is validated with well-established experimental, analytical and numerical results. The algorithm developed for analyzing viscous flow through ducts is extended with slight modification for flow through circular annuli. In fact a circular duct is a special case of an annulus where the inner to outer wall ratio becomes zero and the axis of symmetry becomes the line along which the axial component of velocity is maximum. The various results associated with flow through an annulus are reproduced from Bird et al. [10]. Axial component of velocity

$$u = \frac{(P_0 - P_L)R^2}{4\pi L} \left[ 1 - \left(\frac{r}{R}\right)^2 + \frac{1-K^2}{\ln\left(\frac{1}{K}\right)} \ln\left(\frac{r}{R}\right) \right] \quad \dots(4.1)$$

$$u_{\max} = \frac{(P_0 - P_L)R^2}{4\pi L} \left[ 1 - \left( \frac{1-K^2}{2\ln\left(\frac{1}{K}\right)} \right) \left( 1 - \ln \left\{ \frac{1-K^2}{2\ln\left(\frac{1}{K}\right)} \right\} \right) \right] \quad \dots(4.2)$$

In non-dimensional form this equation can be expressed as,

$$u = c \left[ 1 - r^2 \right] + \left[ \frac{1-k^2}{\ln\left(\frac{1}{k}\right)} \right] \ln r \quad (4.3)$$

c can be shown to be a function of K only and,

$$c = \frac{2 \ln K}{(1+K^2)\ln K + (1-K^2)} \quad \dots(4.4)$$

Thus the analytical values of velocity across a section beyond the entrance length is a function of K only.

The velocity profile for the annular ducts obtained numerically and analytically have been compared in table 1, for k=0.1. As the numerical results have been presented for a wide Reynolds number range, their comparison with the analytical results clearly established the algorithm developed to be quite powerful. The developing velocity profiles in the annular duct for Reynolds number of 1000 have been shown in Fig 2 and 3, for K=0.1 and K=0.3. Fig 4 shows only the fully developed velocity profiles for K=0.4, 0.6 and 0.8. In these figures the radial direction

has been transformed to a new coordinate defined as

$$\eta = \frac{r-R}{1-K}$$

It is found in these figures that the velocity profile is not symmetric in the annular space. The location of the peak velocity can be found out by differentiating Eq. (4.2) and setting it to zero. This gives that for K=0.1 and K=0.2 the corresponding locations of peak velocities are given respectively by  $\eta = 0.404$  and  $\eta=0.430$ . These findings are in excellent agreement with the numerically obtained values. The peak velocity locations corresponding to K=0.4, 0.6 and 0.8 as revealed by Fig. 4 are also in close agreement with the analytical results of  $\eta=0.46, 0.47$  and  $0.45$  respectively. These results also show that lesser is the value of k more inclined is the location of the peak velocity towards the inner wall. The axi-symmetric form of the velocity profile can be attributed to the unequal growth pattern of the boundary layers at the inner and outer walls, which has been corroborated also by the velocity distribution at intermediate stations within the entrance region.

Mohanty and Asthana [8] were the first to announce that the velocity profile needs a substantial axial distance to attain the Poiseuille parabolic shape, even after the boundary layer have merged on the centerline, for through circular pipes. They reported that this merger takes place at 25% of the entrance length. This phenomenon has not been checked for other flow geometries earlier and so similar computations were carried out for annular ducts. Table 2 presents relevant values for K=0.1 and 0.5. It is seen that the ratio increases with K and infact for k=0.5 the value is 0.85.

## CONCLUSIONS

A through study on the hydrodynamic developing zone is carried out for flow through circular annulus. The following conclusions can be drawn:

The boundary layers coalesce much earlier than the velocity profile is fully developed. For annuli the effects are less pronounced as the inner to outer wall ratio is increased.

## NOMENCLATURE

- C- Pressure drop coefficient
- K- Inner diameter to outer diameter ratio
- Re- Reynolds number
- u-velocity in X-direction
- $u_0$ - Average velocity
- R- Outer radius of the duct
- v- velocity in Y direction
- $\Psi$  -Stream function
- $\omega$  -Vorticity
- $\nu$  -Kinematics viscosity
- $\eta$  - Modified radial coordinate for annulus

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**Table 1 Comparison of numerical and analytical velocity profiles for annulus with K=0.1**

Radial distance From inner wall	Analytical results	Numerical results at Reynolds no. Of				
		10	100	500	1000	2000
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.2	1.29	1.30	1.29	1.29	1.29	1.30
0.4	1.27	1.57	1.58	1.58	1.58	1.58
0.6	1.36	1.38	1.38	1.38	1.38	1.38
0.8	0.83	0.84	0.85	0.84	0.84	0.84
1.0	0.00	0.00	0.00	0.00	0.00	0.00

**Table 2 Comparison of distance indicating boundary layer merger with entrance length for annular due.**

Reynolds No.	Boundary layer Merger at		Entrance length		Ratio	
	K=0.1	K=0.5	K=0.1	K=0.5	K=0.1	K=0.5
100	0.62	0.45	0.95	0.52	0.65	0.87
500	3.01	2.17	4.75	2.56	0.63	0.85
1000	6.21	4.58	9.5	5.39	0.65	0.85
2000	12.16	8.63	90	10.27	0.64	0.84

**Table 3 Comparison of the distance indicating boundary layer merger with entrance length for circular ducts**

Reynolds No.	Boundary layer Merger at	Entrance length	Ratio
100	2.08	6.5	0.32
500	9.92	31.09	0.32
1000	21.44	67.34	0.32
2000	41.60	130.21	0.32